On Being First, Being Wrong and Being Right
Knuth, "Knuth", Wu Wenjun, and Algorithms

Jens Høyrup
December 2018

Manuscript for conference contribution
27 December 2018

Before approaching my topic I need to make clear that I do not read Chinese. What I deal with is thus a topic that can be approached through languages I am familiar with – and indeed, through English and French.

My starting point is a passage in Jiri Hudecek's political biography of Wu Wenjun [Hudecek 2014: 117*f*]:

> although Greek geometry was axiomatized, axiomatization was not its working method in the same way as in modern mathematics. Perhaps because of these problems, in his later works Wu introduced an opposition between proofs and algorithms as a replacement.[1] A crucial influence in this shift was the work of the computer scientist Donald Knuth. Wu studied Knuth's textbook *The Art of Computer Programming* (first volume 1968, second 1969, third 1973), which consists of commented algorithms, just like ancient Chinese mathematical classics consisting of problem-solving methods. Knuth also wrote an article on ancient Babylonian algorithms (Knuth 1972). Although never cited by Wu himself, it was mentioned by his younger colleagues Li Wenlin and Yuan Xiandong (1982) in the same volume as Wu Wen-Tsun (1982c). Both Li Wenlin and Karine Chemla, who studied ancient Chinese mathematics in Beijing in the early 1980s, confirmed the influence of this article on Wu Wen-Tsun's thought (personal communication).
>
> Wu might as well have drawn inspiration from Knuth's opening sentence: "One of the ways to help make computer science respectable is to show that it is deeply rooted in history, not just a short-lived phenomenon." Wu Wen-Tsun had a similar motivation for his own turn to history, although it could also be said that he proceeded in the opposite direction, making ancient Chinese mathematics respectable by showing what can be rooted in it.
>
> Knuth drew a series of analogies between ancient Babylonian mathematics and computer science: Babylonian sexagesimal notation was actually the first floating-point notation; their algebraic algorithms were "machine language" as opposed to the "symbolic language" of our modern algebra; they used numerical algebra disregarding physical and geometrical significance. Knuth also compared particular algorithms to a "stack machine" or to a "macro expansion". His article was not a serious history of algorithms, but rather a reminder of the venerable ancestry of the basic techniques of computer science. But Knuth's last paragraph must have been very suggestive to Wu:
>
>> What about other developments? The Egyptians were not bad at mathematics, and archaeologists have dug up some old papyri that are almost as old as the Babylonian tablets. The Egyptian method of multiplication, based essentially in the binary number system (...) is especially interesting. Then came the Greeks, with an emphasis on geometry but also such things as Euclid's algorithm; the latter is the oldest nontrivial algorithm which is still important to computer programmers. (...) And then there are the Indians, and the Chinese; it is clear that much more can be told.
>>
>> (Knuth 1972: 676).

[1] [Namely of the dichotomy axiomatization–mechanization, which Hudecek and perhaps Wu sees/saw as "not quite satisfactory from the historical perspective". /JH]

> In comparison to Knuth's article, Wu's sole emphasis on Chinese mathematics appears narrow-minded. It might even be suggested that Wu tried to take away some credit from "rival" ancient civilizations in his later attempts to demonstrate the computational superiority of specific Chinese algorithms over Western ones.

I have no opinion whether Knuth's *Art of Computer Programming* was a crucial influence, and it is irrelevant to my topic; in any case it seems to regard only the terminological shift from "mechanization" to "algorithms".[2] The confirmation of "the influence [of Knuth's algorithmic interpretation of Babylonian mathematics] on Wu's thought arouses my doubts. Firstly, according to a number of quotations in [Hudecek 2014: 116–119], Wu was no miser when it came to recognizing his debts. Secondly, he had strong doubts concerning the attribution of, for instance, "algebra" to the ancient Babylonians,[3] and also concerning Dirk Struik's belief in the existence of an undifferentiated "Oriental" mathematics [Struik 1948: I, xii and *passim*]. Knuth, on the other hand, had no doubts as to the authenticity of Babylonian algebra, and also suggest is in the passage quoted by Hudecek that ancient Babylonian, Egyptian, Indian and Chinese mathematics belonged to a shared genre. Wu may well have known [Knuth 1972] at some moment, but in that case he seems to have followed the principle formulated by my old friend Marinus Taisbak in a private letter, "in the interest of peace on earth not to cite those with whom I disagree". In any case, his autobiographical note [Wu 2017] suggests that his insight in the algorithmic (at first named "mechanical") nature of ancient Chinese mathematics must antedate his supposed encounter with Knuth's article:

> During the cultural revolution I was sent to a factory manufacturing computers. I was initially struck by the power of the computer. I was also devoted to the study of Chinese ancient mathematics and began to understand what Chinese ancient mathematics really was. I was greatly struck by the depth and powerfulness of its thought and its methods. It was under such influence that I investigated the possibility of proving geometry theorems in a mechanical way.

---

[2] Capriciously, I come to think of Charles Darwin's words [1872: 49] from the sixth edition of the *Origin of Species* concerning Herbert Spencer's "survival of the fittest" – namely that the expression "is more accurate, and is sometimes equally convenient". Of course, this ambiguous praise had no impact on Darwin's own theory, and is not even found worthy a reference in the index.

[Hudecek 2012: 55], speaking of Wu's endeavour to make use of mechanization in contemporary production of mathematical knowledge, only states that "Wu perhaps saw a positive example in the works of the computer scientist Donald Knuth". No claim of crucial influence here.

[3] Had he had access to Neugebauer's or Thureau Dangin's text editions he might *perhaps* have been less sanguine on this account. But this is hypothetical history, only pertinent by emphasizing that he did not have this access.

Babylonia

Let us go on with a closer look at what Knuth says about ancient algorithms, firstly in order to judge the quality of his arguments in their chronological context, secondly and briefly (and somewhat anachronistically) from an actual point of view.

As formulated by Hudecek, Knuth's paper was no piece of serious history writing (neither of algorithms nor otherwise, one may add); nor was it meant to be. In Knuth's own opening words [1972: 671],

> One of the ways to help make computer science respectable is to show that it is deeply rooted in history, not just a short-lived phenomenon. Therefore it is natural to turn to the earliest surviving documents which deal with computation, and to study how people approached the subject nearly 4000 years ago.

Knuth's central argument consists in the presentation of some stepwise calculations in Neugebauer's translation (the best available at the time – somewhat straightened by Knuth in the interest of readability), in which the texts seem to prescribe a sequence of purely arithmetical steps. These are taken to illustrate (p. 672) that

> The Babylonian mathematicians were [...] adept at solving many types of algebraic equations. But they did not have an algebraic notation that is quite as transparent as ours; they represented each formula by a step-by-step list of rules for its evaluation, i.e. by an algorithm for computing that formula. In effect, they worked with a "machine language" representation of formulas instead of a symbolic language.

At the same time Knuth complains (p. 674) that his examples represent

> only "straight-line" calculations, without any branching or decision-making involved. In order to construct algorithms that are really nontrivial from a computer scientist's point of view, we need to have some operations that affect the flow of control.

Knuth overlooks that this linearity is exactly what allows him to conflate the single calculation – by necessity unbranched – with an algorithm which it is supposed to represent.[4] In order to get something which smacks of a loop with a criterion for when to stop Knuth then points to problems about composite interest, where (thus Knuth, p. 674) a "longwinded and rather clumsy procedure reads almost like a macro expansion" – which it evidently only does in the eyes of somebody familiar with macros. The Babylonian calculator simply repeats the calculation.

Knuth also observes (p. 674) that

---

[4] *If* the calculations corresponded to an actual branched algorithm, they might of course justify the road taken by a reference to the criterion deciding what to do at the branching point. The language for that was at hand – regularly, texts justify a step with the words "because he [the master formulating the problem] has said; but no such justification ever corresponds to a branching.

> We don't find tests like "Go to step 4 if x < 0", because the Babylonians didn't have negative numbers; we don't even find conditional tests like "Go to step 5 if x = 0", because they didn't treat zero as a number either! Instead of having such tests, there would effectively be separate algorithms for the different cases. (For example, see [MKT I, 312-314] for a case in which one algorithm is step-by-step the same as another, but simplified since one of the parameters is zero.)

Nor do we find anything like "Go to step 4 if x > 10" – the absence of zero and negative numbers are no reason that the Babylonian texts contain no decisions or branchings. They simply do not. That there are "separate algorithms for the different cases" illustrates instead that Knuth's algorithm concept is empty when applied to the Babylonian record, and hides the genuine character of the texts. The Babylonian texts contain examples meant to be paradigmatic but also meant to be applied with the necessary flexibility.[5] Some texts even show that different approaches are possible – thus the Old Babylonian text AO 8862 [MKT I, 108–117], whose first three problems *could* be solved by application of the same method; instead of doing that, the text applies three different tricks.

What Knuth could not know is that the Babylonian texts do *not* "prescribe a sequence of purely arithmetical steps", as they do in Neugebauer's and Thureau-Dangin's translations; the so-called "algebra" texts prescribe cut-and-paste manipulations of areas within a square-grid geometry, in which the correctness of procedures is as intuitively obvious as in simple symbol-based equation algebra – see, for instance, [Høyrup 2017]. But this is a different matter which does not concern us here. What Knuth *could* have known in 1972 is that the "zigzag functions" describing planetary motion in the Babylonian mathematical astronomy of the later first millennium BCE must have been calculated according to fixed algorithms (with branchings) – see [ACT I, 30–32]. The planetary tables do not explain these algorithms, it is true, they only state what comes out of them. However, the astronomical "procedure texts" explain indubitable algorithms, often with decision of the type DO ... WHILE; a number of (mostly fragmentary) examples are found in [ACT I, 186–276]. A fairly well-conserved complete specimen was published by Lis Brack-Bernsen and Hermann Hunger in [2008]. In the late period, the Babylonian astronomer-priest who prepared these texts were thus fully able to think algorithmically – but that does not entail that the scribe-school teachers did so 1200 to 1700 years earlier.[6]

Nor can the passage [Knuth 1972: 676]

[5] Whoever is tempted to introduce the notion of a "flexible algorithm" (Knuth did not!) should try to write a "flexible" computer program without making use of explicit branchings etc. Algorithms are, by definition, *not flexible* – they are, in Wu's original formulation, *mechanical* [Hudecek 2014: 117*f*].

[6] The question of algorithms in Babylonian mathematics as it can be judged today is dealt with in more detail in [Høyrup 2018].

> What about other ancient developments? The Egyptians were not bad at mathematics, and archeologists have dug up some old papyri that are almost as old as the Babylonian tablets we have discussed. The Egyptian method of multiplication, based essentially in the binary number system (although their calculations were decimal, using something like Roman numerals) is especially interesting; but in other respects, their use of awkward "unit fractions" left them far behind the Babylonians [...]. And then there are the Indians, and the Chinese; it is clear that much more can be told

be counted as arguments that Ancient Egyptian, Indian and Chinese mathematics contained algorithms. Nor can it be supposed that Knuth had more information which he chose not to present in detail. Apart from the Babylonian text collections [MCT, MKT, TMB], the only items in his bibliography that speak of history are Asger Aaboe's *Episodes from the Early History of Mathematics* [1964], a high-school book; and two popularizations, Neugebauer's *Exact Sciences in Antiquity* [1957] and B. L. van der Waerden's *Science Awakening* [1954]; none of these would have assisted him.[7]

Actually, Knuth was too wise to make the claim. This closing passage points back to the beginning of the article, with the exhortation to "turn to the earliest surviving documents which deal with computation, and to study how people approached the subject nearly 4000 years ago". The "Knuth" compared to whom Wu is found by Hudecek to be narrow-minded is a product of superficial reading.

Ancient China

Let us look now at ancient Chinese mathematics, Wu's case. Evidently, a text like the *Nine Chapters* prescribes calculations and can thus be claimed to "contain algorithms" if read as Knuth reads his Mesopotamian material. That seems to be the level at which Hudecek seems to understand him – at least, what [Hudecek 2014: 130] explains suggests nothing else. However, Wu's reference to his "being struck by the depth and powerfulness" of ancient Chinese mathematics shows – even to the one who cannot read the detailed arguments of his Chinese publications – that Wu moves at a different level. Knuth, indeed, when he tried to legitimize modern computer science, had complain about the shallowness of the Babylonian straight-line calculations; that leaves space for neither depth nor particular powerfulmess.

Firstly, there is the way numerical procedures such as the extraction of square and cube roots are presented in the *Nine Chapters*. As formulated by Karine Chemla [1991: 75], (summarizing earlier work from her hand), these sometimes make "use of iteration, conditionals and assignments of variables: three resources listed as basic concepts in D. Knuth's *Fundamental Algorithms. The Art of Computer Programming*". And, in contrast to the conditionals etc. which Knuth imposed on the Mesopotamian material, these

[7] A different question is whether algorithmic analysis can be applied to Mesopotamian and Pharaonic mathematics. Jim Ritter [2004] and Annette Imhausen [2003] have shown convincingly that it can. But none of them claim that the material they deal with consisted *in itself* of algorithms.

are really in the texts – see the appendix in [Chemla 1987], offering text translations and a flow chart corresponding to a critical passage.

I shall not go in detail with this – Chemla has treated the topic under various perspectives immensely better than I would be able to, and not only because she has access to the original texts. Instead, I shall move from the question of *algorithms* to that of *algorithmic culture*, which is rather what Wu speaks about though in different terms.

As Hudecek paraphrases or interprets Wu, "although Greek geometry was axiomatized, axiomatization was not its working method". This "working method" refers to the way axiomatization unfolded from the outgoing 19th century onward and symbolized by Bourbaki ("whom" Wu knew well from proper professional experience). In a different sense, a culture of axiomatization developed in Greek geometry during the fourth century BCE, becoming hegemonic ideology from Euclid's time onward [Høyrup 2018]. That was preceded by a mathematical culture which still saw argument as an ideal, but where arguments would draw on the "locally obvious", with no reference to absolute first principles.[8]

Babylonian mathematical culture was different. It might also appeal to the locally obvious in didactical explanations, but on the whole implicitly (however, oral didactical expositions may have been more explicit). The main teaching aim, as mentioned, was to train through paradigmatic examples, meant to be followed with as much flexibility as needed – what Knuth sees as "separate algorithms for the different cases".

And ancient Chinese mathematical culture was different from all of these – its ideal, even when meant to convey understanding, was *algorithmic*. This may be illustrated by a look at Chapter 3 of the *Nine Chapters* [ed., trans. Chemla & Guo 2004: 280–311]. The chapter deals with distribution according to "degrees" – as explained by Liu Hui, according to *rank*.

Such distributions were well known in Pharaonic Egypt – one example is in Rhind Mathematical Papyrus no. 63 [trans. Peet 1923: 107]. The rule is also the same. Further, the problem type is very common in late medieval and Early Modern European commercial arithmetic, where it goes under the name of the "rule of company" (and similarly); even here the technique is the same. There is a difference, however, and that reflects the difference between mathematical cultures. The *Nine Chapters* at first sets out a rule in abstract terms – an (unbranched) *algorithm*. The follow a number of examples, which are really *applications* of the rule. This is a first hint that the mathematical culture of the *Nine Chapters* is *algorithmic*.

In this sense, even Sanskrit mathematics between Brahmagupta and Bhaskara is often (not always) algorithmic. But there is more to the *Nine Chapters*, Chapter 3. In

[8] Hippocrates of Chios thus bases his investigation of lunules on two principles – the "Pythagorean rule" and the proportionality of areas to the square of a characteristic linear dimension. Both principles had been used in practical mensurational geometry at least since the earliest second millennium BCE. See [Høyrup 2019].

problems no. 1, 3 and 5, the weights are immediately given, and the rule can be applied as it is.[9] In no. 2, however, the three weights are not given directly, but they are told to be in ratios 1:2 and 1:2 – and in no. 4, 5 weights are in ratios 1:2, 1:2, 1:2, 1:2 and 1:2. Obviously, the weights have to be calculated first, as 1–2–4 and 1–2–4–8–16, respectively. However, this preliminary calculation is not explained in either case, its result is stated directly. That is, *the text only explains that part which is covered by the algorithm* as explained in the beginning of the chapter; what falls outside the algorithms also falls outside explanation. *Teaching the algorithm* is thus what the text is about – the higher-level aim of teaching how to make calculations is not directly visible but mediated through the algorithms.

This centrality of the algorithm does not characterize the *Nine Chapters* alone. This work, though a cardinal classic, after all did not constitute a mathematical culture. Ancient Chinese mathematical culture was *a practice*, in which use of the classic was important. But the two should not be conflated. However, a description of state examinations in mathematics written almost a millennium after the *Nine Chapters* (in the *Xin Tang shu*, compiled in 1060) specifies that one of the tasks the students have to perform is to *construct algorithms* (quotations in [Siu & Volkov 1990: 92*f*]. Commentaries (like that of Liu Hui) also explain why algorithms work, and the *creation of new algorithms* (not unspecifally of new mathematical knowledge) is something of which several authors boast (quotations *ibid.* p. 94). So, just as Greek-style mathematics from Euclid to modern times has tended to see the discovery of theorems[10] as the gist of the undertaking, ancient Chinese mathematics saw its task as constructing algorithms – obviously, again, demonstrably correct and coherent algorithms.

Even in this sense, ancient Chinese mathematics, from Han to Tang, was thus *algorithmic*. And in this sense, neither Sanskrit nor Mesopotamian nor Pharaonic (nor medieval Arabic and European) mathematics was.

In consequence, the attentive outsider must conclude that Wu was right – not only in his characterization of ancient Chinese mathematics but also when taking this as the *specific* character of ancient Chinese mathematics.

The same outsider (but now an insider) must conclude that Knuth was mistaken – and even more mistaken the reproach that Wu saw matters more narrowly than "Knuth".

Under these conditions, it is rather futile to discuss who was first. No doubt Baron von Münchhausen was first when it came to riding through the air (on a cannon ball), and the Wright Brothers second.

---

[9] The weights are not always ranks, but Liu Hui has explained already in his commentary to the rule how to apply it if the "degrees" represent the number of members of families, where the distribution is meant to be equal between individuals.

[10] Obviously with appurtenant demonstrations, without which we usually speak of *conjectures* and not of theorems, unless the inventor is a Fermat.

References

Aaboe, Asger, 1964. *Episodes from the Early History of Mathematics*. New York: Random House.

ACT: Otto Neugebauer, *Astronomical Cuneiform Texts: Babylonian Ephemerides of the Seleucid Period for the Motion of the Sun, the Moon, and the Planets*. London: Lund Humphries, 1955.

Brack-Bernsen, Lis, & Hermann Hunger, 2008. "BM 42484+42294 and the Goal-Year method". *SCIAMUS* **9**, 3–23.

Chemla, Karine, & Guo Shuchun (eds), 2004. *Les neuf chapitres. Le Classique mathématique de la Chine ancienne et ses commentaires*. Paris: Dunod.

Chemla, Karine, 1987. "Should They Read Fortran As If It Were English?" *Bulletin of Chinese Studies* **1**, 301–316.

Chemla, Karine, 1991. "Theoretical Aspects of the Chinese Algorithmic Tradition (First to Third Centuries)". *Historia Scientiarum* **42**, 75–98.

Høyrup, Jens, 2017. *Algebra in Cuneiform: Introduction to an Old Babylonian Geometrical Technique*. Berlin: Edition Open Access.

Høyrup, Jens, 2018. "Was Babylonian Mathematics Algorithmic?", pp. 297–312 *in* Kristin Kleber, Georg Neumann & Susanne Paulus (eds), *Grenzüberschreitungen: Studien zur Kulturgeschichte des Alten Orients*. Festschrift für Hans Neumann zum 65. Geburtstag am 9. Mai 2018. Münster: Zaphon, 2018.

Høyrup, Jens, 2019. "From the Practice of Explanation to the Ideology of Demonstration: an Informal Essay". Forthcoming in Gert Schubring (ed.), *Interfaces between Mathematical Practices and Mathematical Education*. New York: Springer, 2019.

Hudecek, Jiri, 2012. "Ancient Chinese Mathematics in Action: Wu Wen-Tsun's Nationalist Historicism after the Cultural Revolution". *East Asian Science, Technology and Society* **6**, 41–64.

Hudecek, Jiri, 2014. *Reviving Ancient Chinese Mathematics: Mathematics, History and Politics in the Work of Wu Wen-Tsun*. London & New York: Routledge.

Imhausen, Annette, 2003. *Ägyptische Algorithmen. Eine Untersuchung zu den mittelägyptischen mathematischen Aufgabentexten*. Wiesbaden: Harrassowitz.

Knuth, Donald E., 1972. "Ancient Babylonian Algorithms". *Communications of the Association of Computing Machinery* **15**, 671–677, with correction of an *erratum* in **19** (1976), 108.

MCT: Otto Neugebauer & Abraham Sachs, *Mathematical Cuneiform Texts*. New Haven, Connecticut: American Oriental Society.

MKT: Otto Neugebauer, *Mathematische Keilschrift-Texte*. 3 vols. Berlin: Julius Springer, 1935, 1935, 1937.

Neugebauer, Otto, 1957. *The Exact Sciences in Antiquity*. Second edition. Providence, Rh.I.: Brown University Press.

Peet, T. Eric, 1923. *The Rhind Mathematical Papyrus, British Museum 10057 and 10058*. Introduction, Transcription, Translation and Commentary. London: University Press of Liverpool.

Ritter, Jim, 2004. "Reading Strasbourg 368: A Thrice-Told Tale", pp. 177–200 *in* Karine Chemla (ed.), *History of Science, History of Text*. Dordrecht: Kluwer.

Siu, Man-Keung, & Alexeï Volkov, 1999. "Official Curriculum in Traditional Chinese Mathematics: How Did Candidates Pass the Examinations?" *Historia Scientiarum* **9**, 85–99.

Struik, Dirk J., 1948. *A Concise History of Mathematics*. 2 vols. New York: Dover.

TMB: François Thureau-Dangin, *Textes mathématiques babyloniens*. Leiden: Brill.

van der Waerden, B. L., 1954. *Science Awakening*. Groningen: Noordhoff.
Wu, Wenjun, 2017. “Autobiography of Wentsun Wu (1919–2017” [written 2006]. *Notices of the AMS* **64**:11, 1319–1320.